Example 7.6:

iii) Comute $\int \arcsin x\,dx$, $(-1<x<1)$.

$$\int \arcsin x\,dx = x\arcsin x - \int x\,d\arcsin x.$$

Now

$$\int x\,d\arcsin x = \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$(t = 1-x^2,\ dt = -2x\,dx)$$

$$= -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2},$$

therefore

$$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2}$$

Another method to compute $\int x\,d\arcsin x$ is to do the substitution $t = \arcsin x$:

$$\int x\,d\arcsin x = \int \sin t\,dt = -\cos t$$

$$= -\sqrt{1-\sin^2 t} = -\sqrt{1-x^2}$$

(We have $\cos t \geqslant 0$, as $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$)

iv) Let $t \neq 0$ be a real parameter. Applying partial integration twice, we obtain

$$\int e^{tx}\sin x\,dx = \frac{1}{t}e^{tx}\sin x - \frac{1}{t}\int e^{tx}\cos x\,dx$$

$$= \frac{1}{t} e^{tx} \sin x - \frac{1}{t^2} e^{tx} \cos x - \frac{1}{t^2} \int e^{tx} \sin x \, dx .$$

Solving for $\int e^{tx} \sin x \, dx$, we obtain

$$\int e^{tx} \sin x \, dx = \frac{e^{tx}}{1+t^2} (t \sin x - \cos x)$$

<u>Example 7.7:</u>

Using partial integration, one can often derive recursion formulas for integrals depending on an integer number.

i) Consider for $m \geqslant 1$ the integral

$$I_m := \int \frac{dx}{(1+x^2)^m}$$

Partial integration gives

$$\int \frac{1}{(1+x^2)^m} dx = \frac{x}{(1+x^2)^m} - \int x \, d\left(\frac{1}{(1+x^2)^m}\right)$$

$$= \frac{x}{(1+x^2)^m} + 2m \int \frac{x^2}{(1+x^2)^{m+1}} dx$$

$$= \frac{x}{(1+x^2)^m} + 2m \int \frac{dx}{(1+x^2)^m} - 2m \int \frac{dx}{(1+x^2)^{m+1}}$$

$$\Rightarrow 2m I_{m+1} = (2m-1) I_m + \frac{x}{(1+x^2)^m}$$

As $I_1 = \arctan x$, one can deduce from this all I_m for $m \geq 1$. In particular, one obtains

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\left(\arctan x + \frac{x}{1+x^2}\right)$$

which one can verify by differentiating the right-hand side.

ii) Consider now the integrals

$$I_m := \int \sin^m x \, dx$$

Partial integration gives for $m \geq 2$

$$I_m = -\int \sin^{m-1} x \, d\cos x$$

$$= -\cos x \sin^{m-1} x + (m-1)\int \cos^2 x \sin^{m-2} x \, dx$$

$$= -\cos x \sin^{m-1} x + (m-1)\int (1-\sin^2 x)\sin^{m-2} x \, dx$$

$$= -\cos x \sin^{m-1} x + (m-1) I_{m-2} - (m-1) I_m .$$

Solving this equation for I_m gives

$$I_m = -\frac{1}{m}\cos x \sin^{m-1} x + \frac{m-1}{m} I_{m-2} .$$

As $I_0 = \int \sin^0 x \, dx = x$, $I_1 = \int \sin x \, dx = -\cos x$, one can this way compute all I_m recursively.

iii) Now let us look at the following definite integral

$$A_m := \int_0^{\pi/2} \sin^m x \, dx .$$

We have $A_0 = \frac{\pi}{2}$, $A_1 = 1$ and

$$A_m = \frac{m-1}{m} A_{m-2}, \quad \text{for } m \geq 2.$$

One obtains

$$A_{2n} = \frac{(2n-1)(2n-3)\cdot \ldots \cdot 3 \cdot 1}{2n \cdot (2n-2) \cdot \ldots \cdot 4 \cdot 2} \cdot \frac{\pi}{2},$$

$$A_{2n+1} = \frac{2n \cdot (2n-2) \cdot \ldots \cdot 4 \cdot 2}{(2n+1)\cdot(2n-1)\cdot \ldots \cdot 5 \cdot 3} .$$

One direct application of this is Wallis' product representation for π:

<u>Proposition 7.13:</u>

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1}$$

<u>Proof:</u>

As $\sin^{2n+2} x \leq \sin^{2n+1} x \leq \sin^{2n} x$ for $x \in [0, \frac{\pi}{2}]$, we have $A_{2n+2} \leq A_{2n+1} \leq A_{2n}$.

As $\lim_{n\to\infty} \frac{A_{2n+2}}{A_{2n}} = \lim_{n\to\infty} \frac{2n+1}{2n+2} = 1$,

we also have $\lim_{n\to\infty} \frac{A_{2n+1}}{A_{2n}} = 1$.

On the other hand, we know from Example 7.7. iii):

$$\frac{A_{2n+1}}{A_{2n}} = \frac{2n \cdot 2n \cdot \dots \cdot 4 \cdot 2 \cdot 2}{(2n+1)(2n-1) \cdots 3 \cdot 3 \cdot 1} \cdot \frac{2}{\pi}$$

$$= \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \cdot \frac{2}{\pi}.$$

Taking the limit $n \to \infty$ gives the claim. $\square$

Remark 7.4:

Wallis' product is not well-suited for practical computations of π as it converges rather slowly. For example,

$$\prod_{n=1}^{1000} \frac{4n^2}{4n^2-1} = 1.57040\dots,$$

which compared with the exact value

$$\frac{\pi}{2} = 1.5707963\dots$$

is still imprecise.

Proposition 7.14 (Riemann's Lemma):

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuously differentiable function. For $k \in \mathbb{R}$ let

$$F(k) := \int_a^b f(x) \sin kx \, dx$$

Then $\lim\limits_{|k| \to \infty} F(k) = 0$.

Proof:

For $k \neq 0$ we obtain through partial integration

$$F(k) = -f(x) \frac{\cos kx}{k} \Big|_a^b + \frac{1}{k} \int_a^b f'(x) \cos kx \, dx .$$

As f and f' are continuous on $[a,b]$ there exists a constant $M \geq 0$, s.t.

$$|f(x)| \leq M, \text{ and } |f'(x)| \leq M \text{ for } x \in [a,b].$$

From this we obtain

$$|F(k)| \leq \frac{2M}{|k|} + \frac{M(b-a)}{|k|},$$

from which the claim follows. $\square$

Note: The Riemann-Lemma is also valid if f is only R-integrable (leave as exercise)

Example 7.8:

As an example of Prop. 7.14 we show

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \quad \text{for } 0 < x < 2\pi .$$

Proof:

As $\int_{\pi}^{x} \cos kt \, dt = \frac{\sin kt}{k}$ and

$$\sum_{k=1}^{n} \cos kt = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} - \frac{1}{2},$$

we obtain

$$\sum_{k=1}^{n} \frac{\sin kx}{k} = \int_{\pi}^{x} \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} \, dt - \frac{1}{2}(x-\pi)$$

Prop. 7.14 then gives for

$$F_n(x) := \int_{\pi}^{x} \frac{1}{2\sin\frac{1}{2}t} \sin(n+\tfrac{1}{2})t \, dt, \quad (0<x<2\pi)$$

that $\lim_{n\to\infty} F_n(x) = 0$. From this the claim follows. $\square$

Note:

If we insert $x = \frac{\pi}{2}$ into this formula, we get

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \pm \cdots$$

Proposition 7.15 (Trapeze rule):

Let $f:[0,1] \longrightarrow \mathbb{R}$ be a twice continuously differentiable function. Then

$$\int_0^1 f(x)\,dx = \frac{1}{2}\big(f(0)+f(1)\big) - R,$$

where the remainder R is

$$R = \frac{1}{2}\int_0^1 x(1-x) f''(x)\,dx = \frac{1}{12} f''(\xi)$$

for some $\xi \in [0,1]$.

Proof:

Let $\varphi(x) := \frac{1}{2}x(1-x)$. We have $\varphi'(x) = \frac{1}{2} - x$ and $\varphi''(x) = -1$. Partially integrating twice gives

$$\begin{aligned} R = \int_0^1 \varphi(x) f''(x)\,dx &= \varphi(x) f'(x)\Big|_0^1 - \int_0^1 \varphi'(x) f'(x)\,dx \\ &= -\varphi'(x) f(x)\Big|_0^1 + \int_0^1 \varphi''(x) f(x)\,dx \\ &= \frac{1}{2}\big(f(0)+f(1)\big) - \int_0^1 f(x)\,dx \end{aligned}$$

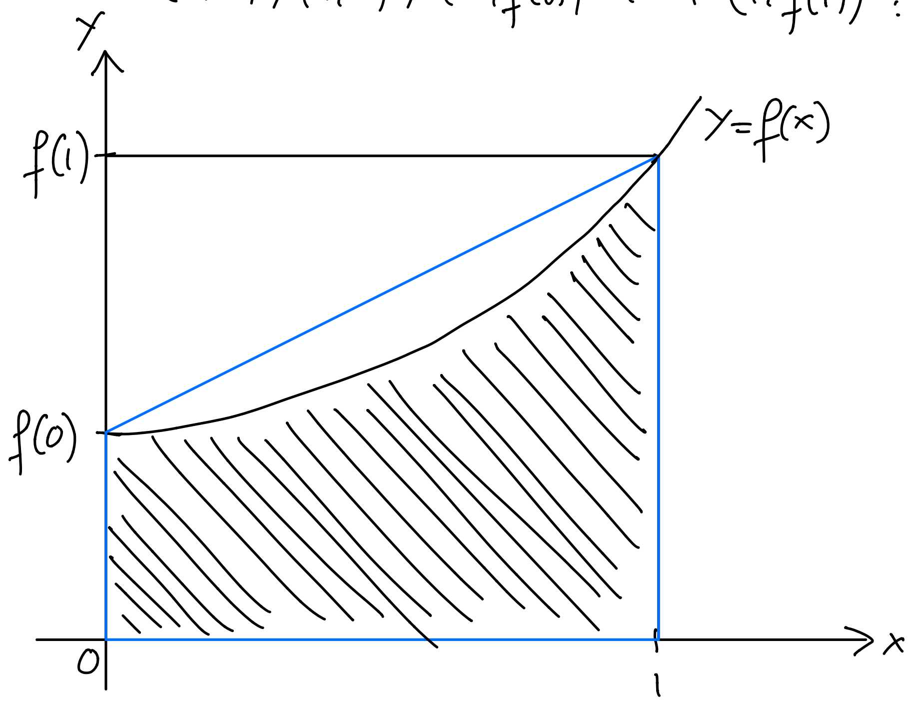
On the other hand, one can apply the mean value theorem (as $\varphi(x) \geq 0 \;\forall\, x \in [0,1]$) and obtains for some $\xi \in [0,1]$:

$$R = \int_0^1 \varphi(x) f''(x)\,dx = f''(\xi) \int_0^1 \varphi(x)\,dx = \frac{1}{12} f''(\xi)$$

$\square$

Remark 7.5:

The name trapeze-rule is due to the fact that the expression $\frac{1}{2}(f(0) + f(1))$ for positive f describes the area of a trapeze with vertices $(0,0)$, $(1,0)$, $(0,f(0))$ and $(1,f(1))$:



One also sees from the above figure why the remainder $-\frac{1}{12} f''(\xi)$ is accompanied by a minus sign: for a convex function ($f'' \geq 0$) the area of the trapeze is bigger than the integral (area under the curve).

Corollary 7.1:

Let $f:[a,b]\longrightarrow \mathbb{R}$ be twice continuously differentiable function and

$$K := \sup\left\{ |f''(x)| \;\middle|\; x\in[a,b]\right\}.$$

Let $n\geqslant 1$ be a natural number and $h:=\frac{b-a}{n}$. Then we have

$$\int_a^b f(x)\,dx = \left(\frac{1}{2}f(a)+\sum_{\nu=1}^{n-1} f(a+\nu h)+\frac{1}{2}f(b)\right)h + R$$

with $|R|\leq \frac{K}{12}(b-a)h^2$.

Proof:

Changing variables, one obtains

$$\int_{a+\nu h}^{a+(\nu+1)h} f(x)\,dx = \frac{h}{2}\big(f(a+\nu h)+f(a+(\nu+1)h)\big) - \frac{h^3}{12}f''(\xi)$$

(Perform substitution $x\mapsto hx$ in Prop. 7.15:

$$\int_0^h f(x)dx = \int_0^1 hf(hx)dx = \frac{h}{2}(f(0)+f(h)) - \frac{h}{12}f''(\xi)h^2\;)$$

with $\xi\in[a+\nu h, a+(\nu+1)h]$. Summation over ν then gives the claim. $\square$

Remark 7.6:

If takes $n \longrightarrow \infty$, the error R goes to 0. This is due to the factor h^2 in the remainder term. The precision is four times accurate if the number of division points is doubled.

Example 7.9:

i) Use the trapezoidal rule to approximate the integral $\int_1^2 \frac{1}{x}\,dx$:

We choose $n=5$, $a=1$, $b=2$

$$\Rightarrow \quad h = \frac{(2-1)}{5} = 0.2$$

Thus the trapezoid rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.2}{2}\Big[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\Big]$$

$$= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

$$\approx 0.695635$$

The exact result is:

$$\int_1^2 \frac{1}{x}\,dx = \ln x\Big|_1^2 = \ln 2 = 0.693147\ldots$$

ii) The continuous function $f(x) = e^{x^2}$ is R-integrable over each interval $[a,b]$, <u>but</u> the indefinite integral is <u>not</u> known!

We evaluate here the integral $\int_0^1 e^{x^2} dx$ using the trapezoidal rule:

Choose $n = 10$, $a = 0$, $b = 1$ $\Longrightarrow$ $h = \frac{1}{10} = 0.1$

Thus
$$\int_0^1 e^{x^2} dx \approx \frac{h}{2}\Big(f(0) + 2f(0.1) + 2f(0.2) + \cdots$$
$$\cdots + 2f(0.9) + f(1)\Big)$$
$$\approx 1.460393$$